Lesson 21. Lagrange Multipliers

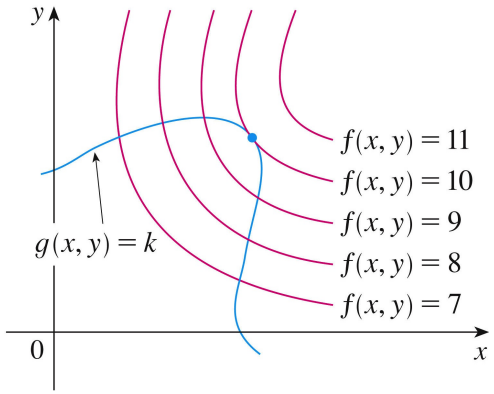
1 In this lesson...

- Optimization with one equality constraint

$$\begin{aligned} \text{minimize or maximize} \quad & f(x,y) \\ \text{subject to} \quad & g(x,y) = k \end{aligned}$$

2 Lagrange multipliers

- Convention: "maximum" and "minimum" refer to "absolute maximum" and "absolute minimum" respectively
- Idea:



  - Maxima and minima occur when the level curves of $f(x,y)$ and the constraint $g(x,y)$ have a common tangent line
  - In other words, the gradients of f and g are parallel:

- **Method of Lagrange multipliers for optimization with one equality constraint**
  - To find the maximum and minimum values of $f(x,y)$ subject to the constraint $g(x,y) = k$:
    1. Find all values of x, y, λ such that

    or equivalently

    2. Evaluate f at all the points (x,y) you found in Step 1.
       - Largest of these values = maximum value of f
       - Smallest of these values = minimum value of f
  - (Assumes extreme values exist and $\nabla g \neq \vec{0}$ on the curve $g(x,y) = k$)
  - Works in a similar way for solving

$$\begin{aligned} \text{minimize or maximize} \quad & f(x,y,z) \\ \text{subject to} \quad & g(x,y,z) = k \end{aligned}$$

Example 1. Find the absolute maximum and minimum of $f(x, y) = y^2 - x^2$ on the ellipse $x^2 + 4y^2 = 4$.

Example 2. Find three positive numbers whose sum is 90 and whose product is a maximum.

Example 3. A rectangular box is to be made from 100 m^2 of cardboard. Find the maximum volume of such a box.

Example 4. Your firm has been asked to design a storage tank for liquid petroleum gas. The customer's specifications call for a rectangular box tank that is to hold 1000 m^3 of gas. (Assume the tank is closed on all sides.) The customer wants to use the smallest amount of material possible in building the tank. What dimensions do you recommend for the tank?

Example 5. You are in charge of erecting a radio telescope on a newly discovered planet. To minimize interference, you want to place it where the magnetic field of the planet is weakest.

If we place the planet in a 3D coordinate system whose origin is at the center of the planet, the surface of the planet can be described by the equation $x^2 + y^2 + z^2 = 36$. The strength of the magnetic field is given by $M(x, y, z) = 6x - y^2 + xz + 60$. Where should you locate the radio telescope?